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ASYMPTOTIC LIMITS FOR MILDLY DEGENERATE KIRCHHOFF EQUATIONS*

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Abstract. We consider the second order Cauchy problem $\varepsilon u_\varepsilon'' + |A^{1/2}u_\varepsilon|^{2\gamma}Au_\varepsilon + u_\varepsilon' = 0$, $u_\varepsilon(0) = u_0 \neq 0$, $u_\varepsilon'(0) = u_1$ where $\varepsilon > 0$, H is a Hilbert space, A is a self-adjoint positive operator on H with dense domain $D(A)$, $(u_0, u_1) \in D(A) \times D(A^{1/2})$, and $\gamma > 0$. We accurately study the decay as t goes to infinity of the solutions, which exist for every ε small enough. In particular we obtain a new estimate on $u_\varepsilon''(t)$ and we show that $(1+t)^{1/2\gamma}Au_\varepsilon(t) \to u_{\varepsilon,\infty} \neq 0$, $(1+t)^{1+1/2\gamma}A^{1/2}u_\varepsilon'(t) \to v_{\varepsilon,\infty} \neq 0$, as t goes to infinity. Moreover we show that the norm of $u_{\varepsilon,\infty}$ and $v_{\varepsilon,\infty}$ does not depend on the initial data.

Key words. degenerate damped hyperbolic equations, Kirchhoff equations, decay rate of solutions

AMS subject classifications. 35B25, 35B40, 35L80

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1. Introduction.

Let H be a real Hilbert space. Given x and y in H, $|x|$ denotes the norm of x and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We always assume that A is coercive, namely, there exists $\sigma_0 > 0$ such that$\langle Au, u\rangle \geq \sigma_0|u|^2$ for every $u \in D(A)$. For any such operator the power A^α is defined for every $\alpha \geq 0$ in a suitable domain $D(A^\alpha)$.

For every $\varepsilon > 0$ we consider the second order Cauchy problem

$$\varepsilon u_\varepsilon''(t) + |A^{1/2}u_\varepsilon(t)|^{2\gamma}Au_\varepsilon(t) + u_\varepsilon'(t) = 0 \qquad \forall\, t \geq 0, \tag{1.1}$$

$$u_\varepsilon(0) = u_0 \neq 0, \qquad u_\varepsilon'(0) = u_1, \tag{1.2}$$

where $u_0 \in D(A)$ and $u_1 \in D(A^{1/2})$. This problem is just an abstract setting (with $m(r) = r^\gamma$) of the initial boundary value problem for the hyperbolic partial differential equation (PDE)

$$\varepsilon u_{tt}^\varepsilon(t,x) - m\left(\int_\Omega |\nabla u^\varepsilon(t,x)|^2\, dx\right)\Delta u^\varepsilon(t,x) + u_t^\varepsilon(t,x) = 0 \tag{1.3}$$

in a bounded open set $\Omega \subseteq \mathbb{R}^n$, where m is a nonnegative function. This equation is a model for the damped small transversal vibrations of an elastic string ($n = 1$) or membrane ($n = 2$) with uniform density ε.

Equations such as (1.1) or (1.3) have been intensely studied in the last 30 years both in the case of coercive operators and in the case of only nonnegative operators. We limit ourselves to the case of coercive operators. For nonnegative operators existence results are similar, but decay estimates are usually worse (see [6]). In the nondegenerate case ($m(\sigma) \geq c > 0$) existence of global solutions for small ε (or equivalently small data) was proved by De Brito [1] and Yamada [20]. In the case of problem (1.1), (1.2) existence of global solutions for small ε was proved by Nishihara

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and Yamada [17] if $\gamma \geq 1$ and by the author (see [3], [4]) if $0 < \gamma < 1$. All these papers also present decay estimates as $t \to +\infty$, which in general are nonoptimal (see also [2], [13], [16] for the nondegenerate case). In the case of (1.1) sharp decay estimates on $|A^{1/2}u_\varepsilon|$, $|Au_\varepsilon|$, $|u'_\varepsilon|$ were obtained by Mizumachi (see [14], [15]) and Ono (see [18], [19]) when $\gamma = 1$ and then for any $\gamma > 0$ by Ghisi and Gobbino (see [6], [7]). We recall such estimates in Theorem 2.1. It is clear that estimates on $|A^{1/2}u_\varepsilon|$, $|Au_\varepsilon|$, $|u'_\varepsilon|$ yield estimates also on u''_ε. These estimates in general are not sharp. Indeed in such a way for $\gamma = 1$ one obtains $(1+t)^3|u''_\varepsilon(t)|^2 \leq C_\varepsilon$ while from [18] we know that at least one has $(1+t)^4|u''_\varepsilon(t)|^2 \leq C_\varepsilon$.

In this paper we study decay estimates from a new point of view (in the same spirit of [11] where, however, an ordinary differential equation is considered). We prove that there exist *nonzero* vectors $u_{\varepsilon,\infty} \in D(A)$, $v_{\varepsilon,\infty} \in D(A^{1/2})$ such that

$$\lim_{t\to+\infty}(1+t)^{1/(2\gamma)}u_\varepsilon(t) = u_{\varepsilon,\infty} \qquad \text{in } D(A), \tag{1.4}$$

$$\lim_{t\to+\infty}(1+t)^{1+1/(2\gamma)}u'_\varepsilon(t) = v_{\varepsilon,\infty} \qquad \text{in } D(A^{1/2}). \tag{1.5}$$

We also compute the norm of $u_{\varepsilon,\infty}$ and $v_{\varepsilon,\infty}$ in $D(A)$ and $D(A^{1/2})$, respectively, and we show that they are independent both of ε and of initial data. On the one hand, (1.4) and (1.5) confirm and strengthen some previous decay estimates (see (2.1)–(2.3) below). On the other hand we obtain also a relation between $u_{\varepsilon,\infty}$ and $v_{\varepsilon,\infty}$ which yields better estimates on u''_ε (see Theorem 3.3). We indeed prove that

$$(1+t)^{4+1/\gamma}|u''_\varepsilon(t)|^2 \leq C_\varepsilon$$

which also improves in the case $\gamma = 1$ the corresponding estimates in [18]. We do not have the analogous estimate from below, but we strongly suspect that the exponent $4 + 1/\gamma$ is optimal because it is the decay rate of u'' for the solutions u of the first order problem

$$u'(t) + |A^{1/2}u(t)|^{2\gamma}Au(t) = 0 \qquad \forall\, t \geq 0, \tag{1.6}$$

$$u(0) = u_0, \tag{1.7}$$

obtained by formally setting $\varepsilon = 0$ in (1.1).

In the following we assume that H has a countable orthogonal system made by eigenvectors of A. This assumption is trivially verified in the case of (1.3) in bounded domains. Under this assumption we have that the norm of $u_{\varepsilon,\infty}$ in $D(A)$ and of $v_{\varepsilon,\infty}$ in $D(A^{1/2})$ depend only on γ and on the smallest eigenvalue of A with respect to which the initial data have nonzero components. Roughy speaking, this is due to the fact that dissipation has a bigger effect on the higher frequencies, so that for t large the component with respect to the smallest frequency is dominant. In other words, when ε is small enough solutions of (1.1) behave as solutions of the first order problem (1.6). The results proved in this paper are fundamental to provide optimal decay-error estimates for the singular perturbation problem, namely, on the difference $u_\varepsilon - u$ between solutions of (1.1) and solutions of (1.6) (see [9], [10]). Indeed in the past such decay-error estimates were known and optimal in the nondegenerate case (see [12]) but only partial results had been proved in the degenerate case (see the survey [8]).

The outline of the paper is the following. In section 2 we fix some notation and we recall the previous results we need concerning the existence of global solutions of (1.1) and their decay at infinity. In section 3 we state the main results. In section 4 we prove the results.

2. Notation and preliminaries. For the convenience of the reader, we recall the following result concerning solutions of (1.1). The existence result follows from [17], [4] (see also [5] for the study of the case of more general functions m). Decay estimates follow from [6], [7] (see also [14], [15], [17], [18], [19]). We stress that the operator is assumed to be coercive.

THEOREM 2.1. *Let A be a self-adjoint coercive operator on a Hilbert space H with dense domain. Let $\gamma > 0$ be a real number and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $u_0 \neq 0$. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, problem* (1.1), (1.2) *has a unique global solution*

$$u_\varepsilon \in C^2([0, +\infty[, H) \cap C^1([0, +\infty[, D(A^{1/2})) \cap C^0([0, +\infty[, D(A)).$$

Moreover there exist positive constants K_1 and K_2 such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that

$$\frac{K_1}{(1+t)^{1/\gamma}} \leq |A^{1/2} u_\varepsilon(t)|^2 \leq \frac{K_2}{(1+t)^{1/\gamma}} \qquad \forall t \geq 0, \tag{2.1}$$

$$\frac{K_1}{(1+t)^{1/\gamma}} \leq |A u_\varepsilon(t)|^2 \leq \frac{K_2}{(1+t)^{1/\gamma}} \qquad \forall t \geq 0, \tag{2.2}$$

$$|u'_\varepsilon(t)|^2 \leq \frac{K_2}{(1+t)^{2+1/\gamma}} \qquad \forall t \geq 0. \tag{2.3}$$

From now on we always assume that $\varepsilon \leq \min\{\varepsilon_0, 1\}$ so that all conclusions of Theorem 2.1 hold true.

Let us now set

$$b_\varepsilon(t) := |A^{1/2} u_\varepsilon(t)|^{2\gamma}, \qquad b_\varepsilon(0) = |A^{1/2} u_0|^{2\gamma} =: b_0. \tag{2.4}$$

An immediate consequence of Theorem 2.1 is that

$$\frac{K_3}{1+t} \leq b_\varepsilon(t) \leq \frac{K_4}{1+t}, \qquad \frac{|b'_\varepsilon(t)|}{b_\varepsilon(t)} \leq \frac{K_4}{1+t} \qquad \forall t \geq 0, \tag{2.5}$$

where the constants K_3 and K_4 do not depend on ε.

Moreover let us define

$$B_\varepsilon(t) := \int_0^t b_\varepsilon(s)\, ds. \tag{2.6}$$

From (2.5) we know that $B_\varepsilon(t) \to +\infty$ as $t \to +\infty$, and we also have a nonoptimal, but ε independent, estimate on its growth rate. Our goal is to obtain sharp estimates on the growth of $B_\varepsilon(t)$ and to exploit them to deduce our estimates on u_ε.

Let us introduce some general notation. Let $(e_k)_k$ be a countable orthogonal system of H made by eigenvectors of A, and λ_k^2 be the corresponding eigenvalues, that is

$$Ae_k = \lambda_k^2 e_k \qquad \forall k.$$

Every $u \in H$ can be written in the form

$$u = \sum_k u_k e_k,$$

where $\{u_k\}$ is the sequence of the components of u with respect to $\{e_k\}$. In particular the solution $u_\varepsilon(t)$ of (1.1), (1.2) can be written as

$$u_\varepsilon(t) = \sum_k u_{\varepsilon,k}(t) e_k,$$

where $u_{\varepsilon,k}(t)$ is the solution of the Cauchy problem

$$\varepsilon u''_{\varepsilon,k}(t) + b_\varepsilon(t)\lambda_k^2 u_{\varepsilon,k}(t) + u'_{\varepsilon,k}(t) = 0, \qquad u_{\varepsilon,k}(0) = u_{0,k},\ u'_{\varepsilon,k}(0) = u_{1,k}. \tag{2.7}$$

For every $\lambda > 0$ let us set

$$H_\lambda := \left\{ u \in H : u = \sum_{k:\ \lambda_k \geq \lambda} u_k e_k \right\}, \qquad H_{\{\lambda\}} := \left\{ u \in H : u = \sum_{k:\ \lambda_k = \lambda} u_k e_k \right\},$$

$$H_{[\lambda,\mu)} := \left\{ u \in H : u = \sum_{k:\ \lambda \leq \lambda_k < \mu} u_k e_k \right\},$$

and let A_λ, $A_{\{\lambda\}}$, $A_{[\lambda,\mu)}$ denote the restriction of the operator A to the subspaces H_λ, $H_{\{\lambda\}}$, $H_{[\lambda,\mu)}$, respectively:

$$A_\lambda = A_{|H_\lambda}, \qquad A_{\{\lambda\}} = A_{|H_{\{\lambda\}}}, \qquad A_{[\lambda,\mu)} = A_{|H_{[\lambda,\mu)}}.$$

Let, moreover, ν be defined by:

$$\nu := \min\{\lambda_k :\ u_{0,k} \neq 0,\ \text{ or } u_{1,k} \neq 0\}. \tag{2.8}$$

We can assume without loss of generality that ν^2 is the smallest eigenvalue of A, so that $A = A_\nu$, and

$$\langle Au, u\rangle \geq \nu^2 |u|^2 \qquad \forall u \in D(A). \tag{2.9}$$

For every $\mu > \nu$ we can decompose any $u \in H$ as

$$u = u_\nu + \overline{u}_\mu + U_\mu, \tag{2.10}$$

where $u_\nu \in H_{\{\nu\}}$ and $U_\mu \in H_\mu$.

Finally, for every $\lambda \geq \nu$ let us define the corrector $\Theta_{\varepsilon,\lambda} \in H_\lambda$ as the solution of

$$\varepsilon\Theta''_{\varepsilon,\lambda} + \Theta'_{\varepsilon,\lambda} = 0, \qquad \Theta_{\varepsilon,\lambda}(0) = 0, \quad \Theta'_{\varepsilon,\lambda}(0) = U_{1,\lambda} + b_0 A_\lambda U_{0,\lambda}. \tag{2.11}$$

3. Main results. We are now ready to state our main results. Our first result concerns the decay of the components of u_ε.

THEOREM 3.1. *Let A be a self-adjoint coercive operator on a Hilbert space H, with dense domain. Let $\gamma > 0$ be a real number and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $u_0 \neq 0$. Let u_ε be the solution of* (1.1), (1.2) *as in Theorem* 2.1. *Let ν defined as in* (2.8) *and let $\lambda \geq \nu$. Then for ε small (depending on λ) we have the following inequalities.*

1. *For $h = 0, 1$ there exist constants $\gamma_{h,\lambda}$ independent of ε such that*

$$e^{2\lambda^2 B_\varepsilon(t)} \left(\varepsilon \frac{|A^{h/2} U'_{\varepsilon,\lambda}(t)|^2}{b_\varepsilon(t)} + |A^{(h+1)/2} U_{\varepsilon,\lambda}(t)|^2 \right) \leq \gamma_{h,\lambda} \qquad \forall t \geq 0. \tag{3.1}$$

2. *There exists a constant γ_λ independent of ε such that*

$$e^{2\lambda^2 B_\varepsilon(t)}\frac{|U'_{\varepsilon,\lambda}(t)|^2}{b_\varepsilon^2(t)} \leq \gamma_\lambda \qquad \forall t \geq 0. \tag{3.2}$$

3. *There exists a constant $\gamma_{\varepsilon,\lambda}$ such that*

$$e^{2\lambda^2 B_\varepsilon(t)}\frac{|U''_{\varepsilon,\lambda}(t) - \Theta''_{\varepsilon,\lambda}(t)|^2}{b_\varepsilon^4(t)} \leq \gamma_{\varepsilon,\lambda} \qquad \forall t \geq 0. \tag{3.3}$$

   If moreover $(u_0, u_1) \in D(A^2) \times D(A^{3/2})$, then also $\gamma_{\varepsilon,\lambda}$ can be taken to be independent of ε.

Remark 3.2. Theorem 3.1 is actually a "linear result," which means that in the proof we use only that all components $u_{\varepsilon,k}(t)$ of $u_\varepsilon(t)$ verify (2.7) with a coefficient $b_\varepsilon(t)$ satisfying (2.5). As a consequence, if initial data are more regular then estimates such as (3.1) through (3.3) hold true also for $A^{h/2}U_{\varepsilon,\lambda}$ with suitable values of h.

Theorem 3.1 says that the components of u_ε with respect to big eigenvalues decay faster than the components with respect to the smallest eigenvalue. How much faster depends on the growth of B_ε. The following result clarifies this aspect.

THEOREM 3.3. *Let A be a self-adjoint coercive operator on a Hilbert space H, with dense domain. Let $\gamma > 0$ be a real number and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $u_0 \neq 0$. Let u_ε be the solution of* (1.1), (1.2) *as in Theorem* 2.1 *and let ν defined as in* (2.8). *Then for ε small there exists a nonzero vector $u_{\varepsilon,\infty} \in H_{\{\nu\}}$ such that as $t \to +\infty$ we have that*

$$(1+t)^{1/(2\gamma)}(u_\varepsilon(t), (1+t)u'_\varepsilon(t)) \to \left(u_{\varepsilon,\infty}, -\frac{1}{2\gamma}u_{\varepsilon,\infty}\right) \quad \text{in} \quad D(A) \times D(A^{1/2}). \tag{3.4}$$

Moreover the following properties hold true.

1. *There exist constants $K_{\varepsilon,1}$, $K_{\varepsilon,2}$ such that*

$$\frac{K_{\varepsilon,1}}{1+t} \leq e^{-2\nu^2\gamma B_\varepsilon(t)} \leq \frac{K_{\varepsilon,2}}{1+t} \qquad \forall t \geq 0. \tag{3.5}$$

   In addition if $u_{0,\nu} \neq 0$, then we can take $K_{\varepsilon,1}$ and $K_{\varepsilon,2}$ independently of ε.

2. *We have that*

$$\lim_{t\to+\infty}(1+t)b_\varepsilon(t) = \frac{1}{2\nu^2\gamma}, \tag{3.6}$$

$$\lim_{t\to+\infty}(1+t)^{1/\gamma}|u_{\varepsilon,\nu}(t)|^2 = \frac{1}{\nu^2(2\nu^2\gamma)^{1/\gamma}}. \tag{3.7}$$

3. *There exists a constant K_ε such that*

$$|u''_\varepsilon(t)|^2 \leq K_\varepsilon \frac{1}{(1+t)^{4+1/\gamma}} \qquad \forall t \geq 0. \tag{3.8}$$

Let us conclude with some comments on Theorem 3.3.

- Let $\lambda_k \geq \nu$ be an eigenvalue of A and let $u_{\varepsilon,k}(t)$ be the relative component of $u_\varepsilon(t)$. From (3.5) and (3.1) it follows that

$$|u_{\varepsilon,k}(t)|^2 \leq \frac{C_\varepsilon}{(1+t)^{\lambda_k^2/(\nu^2\gamma)}}.$$

  From (3.7) it follow that $u_{\varepsilon,\nu}$ decays as $(1+t)^{-1/\gamma}$. This means that the ratio between the decay rate of the two components is λ_k^2/ν^2, and hence the ratio between the corresponding eigenvalues.

- From (3.4) and (3.7) we have that

$$|u_{\varepsilon,\infty}|^2 = \frac{1}{\nu^2(2\nu^2\gamma)^{1/\gamma}}.$$

We also obtain that

$$(1+t)^{1/\gamma}|A^{1/2}u_\varepsilon(t)|^2 \to \frac{1}{(2\nu^2\gamma)^{1/\gamma}}; \tag{3.9}$$

$$(1+t)^{1/\gamma}|Au_\varepsilon(t)|^2 \to \frac{\nu^2}{(2\nu^2\gamma)^{1/\gamma}};$$

$$(1+t)^{2+1/\gamma}|u'_\varepsilon(t)|^2 \to \frac{\nu^2}{(2\nu^2\gamma)^{2+1/\gamma}}, \tag{3.10}$$

$$(1+t)^{2+1/\gamma}|A^{1/2}u'_\varepsilon(t)|^2 \to \frac{\nu^4}{(2\nu^2\gamma)^{2+1/\gamma}},$$

hence the limit of the norms does not depend on initial conditions.
- Limits in (3.9)–(3.10) clarify the estimates in (2.1) and (2.2). They also show that (2.3) is sharp, and that a similar estimate also holds true for $|A^{1/2}u'_\varepsilon|$ (maybe with a constant depending on ε).
- Let ν^2 be the smallest eigenvalue with respect to which initial data have a nonzero component. In the linear case (i.e., when in (1.1) there is a coefficient $b(t)$ satisfying (2.5) in place of $|A^{1/2}u_\varepsilon(t)|^{2\gamma}$) the decay rate of the solution coincides with the decay rate of the component relative to this smallest frequency (all other components decay faster). Moreover the decay rate does depend on ν^2. In the nonlinear case it is once again true that the decay of the solutions is equal to the decay of the smallest frequency component, but the decay rate is independent of ν^2.

4. Proofs. In some of the proofs we employ the following simple comparison result that has already been used in various forms in a lot of papers, starting from [5].

LEMMA 4.1. *Let $f \in C^1([0,+\infty))$ and let us assume that $f(t) \geq 0$ in $[0,+\infty)$, and that there exist two constants $K_5 > 0$, $K_6 \geq 0$ such that*

$$f'(t) \leq -K_5\sqrt{f(t)}\left(\sqrt{f(t)} - K_6\right) \qquad \forall t \geq 0.$$

Then we have that $f(t) \leq \max\left\{f(0), K_6^2\right\}$ for every $t \geq 0$.

We divide the proofs into various parts. First we prove two basic propositions on linear equations. Then we prove Theorem 3.1. After we study the decomposition of u_ε made by (2.10) and finally we prove Theorem 3.3.

4.1. Linear equations and estimates. Let M be a self-adjoint linear operator on H. Let us assume that

$$\langle Mw, w\rangle \geq \sigma_M^2|w|^2 \qquad \forall w \in D(M). \tag{4.1}$$

For $h \geq 0$ let us denote by $|w|_{D(M^h)}$ the norm of the vector w in the space $D(M^h)$.

Let us assume that $b : [0,+\infty[\to]0,+\infty[$ is a C^1 function that verifies

$$b(t) \leq \frac{K_4}{1+t}, \qquad \frac{|b'(t)|}{b(t)} \leq \frac{K_4}{1+t}, \qquad \frac{|b'(t)|}{b^2(t)} \leq \frac{K_4}{K_3} \qquad \forall t \geq 0, \tag{4.2}$$

where K_4 and K_3 are the constants in (2.5). For simplicity in the following we use these notations:

$$\|u\|^2 = |u|^2 \left(1 + b(0)^{-1} + b(0)^{-2}\right) \quad \text{if } u \in H,$$
$$\|u\|^2_{D(M^{h/2})} = |u|^2_{D(M^{h/2})} \left(1 + b(0)^{-1} + b(0)^{-2}\right) \quad \text{if } u \in D(M^{h/2}).$$

Let $v_\varepsilon \in C^2([0,+\infty[, H) \cap C^1([0,+\infty[, D(M^{1/2})) \cap C^0([0,+\infty[, D(M))$ be the solution of the problem

(4.3)
$$\varepsilon v''_\varepsilon(t) + b(t)Mv_\varepsilon(t) + v'_\varepsilon(t) = 0, \qquad v_\varepsilon(0) = v_0 \in D(M), \quad v'_\varepsilon(0) = v_1 \in D(M^{1/2}).$$

Moreover let θ_ε be the solution of

$$\varepsilon\theta''_\varepsilon(t) + \theta'_\varepsilon(t) = 0, \qquad \theta_\varepsilon(0) = 0, \quad \theta'_\varepsilon(0) = v_1 + b(0)Mv_0, \tag{4.4}$$

so that $\theta_\varepsilon(t) = \varepsilon\theta'_\varepsilon(0)(1 - e^{-t/\varepsilon})$, and let us set

$$w_\varepsilon = v_\varepsilon - \theta_\varepsilon.$$

Finally let B be defined as in (2.6) (using $b(t)$ in place of b_ε of course).

Therefore the following propositions hold true.

PROPOSITION 4.2. *Let $h \geq 1$ and let us assume that $(v_0, v_1) \in D(M^{(h+1)/2}) \times D(M^{h/2})$. Then for ε small depending only on σ_M^2, K_3, and K_4 (and not on the initial data or h), for all $t \geq 0$ we have that*

$$e^{2\sigma_M^2 B(t)} \left(\varepsilon \frac{|M^{h/2} v'_\varepsilon(t)|^2}{b(t)} + |M^{(h+1)/2} v_\varepsilon(t)|^2\right) \tag{4.5}$$
$$\leq L_0(\|v_1\|^2_{D(M^{h/2})} + |v_0|^2_{D(M^{(h+1)/2})}) =: L_{h,M},$$
$$e^{2\sigma_M^2 B(t)} \frac{|M^{(h-1)/2} v'_\varepsilon(t)|^2}{b^2(t)} \tag{4.6}$$
$$\leq L_1(\|v_1\|^2_{D(M^{h/2})} + |v_0|^2_{D(M^{(h+1)/2})}) =: H_{h,M},$$

where L_0 and L_1 depend only on σ_M^2, K_3, and K_4.

PROPOSITION 4.3. *Let us assume that $(v_0, v_1) \in D(M^2) \times D(M^{3/2})$. Then for ε small depending only on σ_M^2, K_3, and K_4 and not on the initial data we have that*

$$e^{2\sigma_M^2 B(t)} \frac{|w''_\varepsilon(t)|^2}{b^4(t)} \leq L_2(\|v_1\|^2_{D(M^{3/2})} + |v_0|^2_{D(M^2)}) \qquad \forall t \geq 0, \tag{4.7}$$

where L_2 depends only on σ_M^2, K_3, and K_4.

Now let us prove Proposition 4.2 and Proposition 4.3.

Proof of Proposition 4.2. Let us denote by c_i and C_i various constants that depend only on σ_M^2, K_3, and K_4.

The outline of the proof is the following. First (*Step* 1), we prove, for every $h \geq 0$, that *if we have that*

$$e^{2\sigma_M^2 B(t)} |M^{h/2} v_\varepsilon(t)|^2 \leq R_h \qquad \forall t \geq 0 \tag{4.8}$$

then for all $t \geq 0$ we get that

$$e^{2\sigma_M^2 B(t)} \left[\varepsilon \frac{|M^{h/2} v'_\varepsilon(t)|^2}{b(t)} + |M^{(h+1)/2} v_\varepsilon(t)|^2\right] \tag{4.9}$$
$$\leq 16\sigma_M^2 R_h + C_0(\|v_1\|^2_{D(M^{h/2})} + |v_0|^2_{D(M^{(h+1)/2})}).$$

Since problem (4.3) is linear it is enough to prove this estimate for $h = 0$.

Then (*Step* 2) we show that for $h = 1$ we have (4.8) with

$$R_1 = C_1(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)}). \tag{4.10}$$

Using (4.10) in (4.9) with $h = 1$ we can then conclude that (4.5) holds true if $h = 1$. Since (4.3) is linear, (4.5) will be proved for every $h \geq 1$.

In conclusion, for proving (4.5) we have only to prove (4.9) with $h = 0$ and (4.10).

Finally (*Step* 3) we prove (4.6). Also in this case it is enough to consider the case $h = 1$.

For $\alpha > 0$ let us introduce the following energies that we use in the proofs:

$$\begin{aligned} D_\alpha(t) &:= e^{2\alpha B(t)} \left[\langle \varepsilon v'_\varepsilon(t), v_\varepsilon(t) \rangle + \frac{1}{2}|v_\varepsilon(t)|^2 \right], \\ E_\alpha(t) &:= e^{2\alpha B(t)} \left[\varepsilon \frac{|v'_\varepsilon(t)|^2}{b(t)} + |M^{1/2} v_\varepsilon(t)|^2 \right], \\ F_\alpha(t) &:= e^{2\alpha B(t)} \frac{|v'_\varepsilon(t)|^2}{b^2(t)}. \end{aligned}$$

An easy calculation shows that

$$D'_\alpha = 2\alpha b D_\alpha - b e^{2\alpha B} |M^{1/2} v_\varepsilon|^2 + \varepsilon e^{2\alpha B} |v'_\varepsilon|^2, \tag{4.11}$$

$$E'_\alpha = -e^{2\alpha B} \frac{|v'_\varepsilon|^2}{b} \left(2 + \varepsilon \frac{b'}{b} - 2\alpha\varepsilon b \right) + 2\alpha b e^{2\alpha B} |M^{1/2} v_\varepsilon|^2, \tag{4.12}$$

$$F'_\alpha = -\frac{1}{\varepsilon} F_\alpha \left(2 + 2\varepsilon \frac{b'}{b} - 2\alpha\varepsilon b \right) - \frac{2}{\varepsilon} e^{2\alpha B} \frac{1}{b} \langle v'_\varepsilon, M v_\varepsilon \rangle. \tag{4.13}$$

Step 1: Proof of (4.9) with $h = 0$. Let us choose $\alpha = 2\sigma_M^2 := \alpha_0$.

Estimate on D_{α_0}. We prove that, if ε is small enough, for all $t \geq 0$ we have that

$$\begin{aligned} \int_0^t e^{2\alpha_0 B(s)} b(s) |M^{1/2} v_\varepsilon(s)|^2 ds &\leq |v_1|^2 + |v_0|^2 + C_2 \varepsilon^2 e^{2\alpha_0 B(t)} \frac{|v'_\varepsilon(t)|^2}{b(t)} \\ &\quad + C_3 \varepsilon \int_0^t e^{2\alpha_0 B(s)} \frac{|v'_\varepsilon(s)|^2}{b(s)} ds + 2R_0 e^{\alpha_0 B(t)}. \end{aligned} \tag{4.14}$$

By (4.8) we obtain that

$$\begin{aligned} 2\alpha_0 b D_{\alpha_0} &= 2\alpha_0 \varepsilon b e^{2\alpha_0 B} \langle v'_\varepsilon, v_\varepsilon \rangle + \alpha_0 b e^{2\alpha_0 B} |v_\varepsilon|^2 \\ &\leq \alpha_0 \varepsilon^2 b e^{2\alpha_0 B} |v'_\varepsilon|^2 + 2\alpha_0 b e^{2\alpha_0 B} |v_\varepsilon|^2 \\ &\leq \alpha_0 \varepsilon^2 b^2 e^{2\alpha_0 B} \frac{|v'_\varepsilon|^2}{b} + 2\alpha_0 R_0 b e^{\alpha_0 B}. \end{aligned} \tag{4.15}$$

From (4.11) and (4.15) we therefore get that

$$D'_{\alpha_0} + e^{2\alpha_0 B} b |M^{1/2} v_\varepsilon|^2 \leq \varepsilon(b + \alpha_0 \varepsilon b^2) e^{2\alpha_0 B} \frac{|v'_\varepsilon|^2}{b} + 2\alpha_0 R_0 b e^{\alpha_0 B}. \tag{4.16}$$

Since from (4.2) the function b is bounded by K_4, then integrating (4.16) we arrive at

$$\begin{aligned} \int_0^t e^{2\alpha_0 B(s)} b(s) |M^{1/2} v_\varepsilon(s)|^2 ds &\leq D_{\alpha_0}(0) - D_{\alpha_0}(t) \\ &\quad + \varepsilon c_1 \int_0^t e^{2\alpha_0 B(s)} \frac{|v'_\varepsilon(s)|^2}{b(s)} ds + 2R_0 e^{\alpha_0 B(t)}. \end{aligned} \tag{4.17}$$

Since $\varepsilon \leq 1$ and b is bounded, we can estimate $D_{\alpha_0}(0)$ and $D_{\alpha_0}(t)$ as follows:

$$|D_{\alpha_0}(0)| \leq \varepsilon |v_1||v_0| + \frac{1}{2}|v_0|^2 \leq |v_1|^2 + |v_0|^2, \tag{4.18}$$

$$\begin{aligned} -D_{\alpha_0}(t) &\leq e^{2\alpha_0 B(t)}\left(\varepsilon |v_\varepsilon'(t)||v_\varepsilon(t)| - \frac{1}{2}|v_\varepsilon(t)|^2\right) \\ &\leq \frac{1}{2}\varepsilon^2 e^{2\alpha_0 B(t)}\frac{|v_\varepsilon'(t)|^2}{b(t)} b(t) \leq c_2 \varepsilon^2 e^{2\alpha_0 B(t)}\frac{|v_\varepsilon'(t)|^2}{b(t)}. \end{aligned} \tag{4.19}$$

Plugging (4.18)–(4.19) into (4.17) we achieve (4.14).

Proof of (4.9). Integrating (4.12) and using (4.14) we get that

$$\begin{aligned} E_{\alpha_0}(t) \leq E_{\alpha_0}(0) &- \int_0^t e^{2\alpha_0 B(s)}\frac{|v_\varepsilon'(s)|^2}{b(s)}\left(2 + \varepsilon\frac{b'(s)}{b(s)} - 2\alpha_0 \varepsilon b(s) - 2\alpha_0 C_3 \varepsilon\right) ds \\ &+ 2\alpha_0 C_2 \varepsilon^2 e^{2\alpha_0 B(t)}\frac{|v_\varepsilon'(t)|^2}{b(t)} + 2\alpha_0(|v_1|^2 + |v_0|^2) + 4\alpha_0 R_0 e^{\alpha_0 B(t)}. \end{aligned} \tag{4.20}$$

Thanks to (4.2) we can take ε small enough in such a way that

$$2 - 2\varepsilon \sup_{t\geq 0}\frac{|b'(t)|}{b(t)} - 4\sigma_M^2 \varepsilon \sup_{t\geq 0} b(t) - 4\sigma_M^2 \varepsilon C_3 \geq 1, \tag{4.21}$$

$$4\sigma_M^2 C_2 \varepsilon \leq \frac{1}{2}. \tag{4.22}$$

Plugging (4.21) and (4.22) into (4.20) we obtain that

$$E_{\alpha_0}(t) \leq \frac{|v_1|^2}{b(0)} + |M^{1/2}v_0|^2 + \frac{1}{2}E_{\alpha_0}(t) + 2\alpha_0(|v_1|^2 + |v_0|^2) + 4\alpha_0 R_0 e^{\alpha_0 B(t)}$$

from which

$$\frac{1}{2}E_{\alpha_0}(t) \leq c_3(\|v_1\|^2 + |v_0|^2_{D(M^{1/2})}) + 4\alpha_0 R_0 e^{\alpha_0 B(t)}.$$

Since $\alpha_0 = 2\sigma_M^2$, (4.9) immediately follows dividing all terms by $e^{\alpha_0 B(t)}$.

Step 2: Proof of (4.10). To begin with, let us choose

$$\alpha = \sigma_M^2 - \frac{1}{8K_4} := \beta. \tag{4.23}$$

First we prove that for ε small and $h = 0$, $h = 1$ we have for all $t \geq 0$ that

$$e^{2\beta B(t)}\left(\varepsilon\frac{|M^{h/2}v_\varepsilon'(t)|^2}{b(t)} + |M^{(h+1)/2}v_\varepsilon(t)|^2\right) \leq C_4(\|v_1\|^2_{D(M^{h/2})} + |v_0|^2_{D(M^{(h+1)/2})}) \tag{4.24}$$

and

$$e^{2\beta B(t)}\frac{|v_\varepsilon'(t)|^2}{b^2(t)} \leq C_5(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)}). \tag{4.25}$$

Since (4.3) is linear it is enough to prove (4.24) with $h = 0$.

Estimate on D_β. We prove that, if ε is small enough, for all $t \geq 0$ we have that

$$\int_0^t e^{2\beta B(s)} b(s)|M^{1/2}v_\varepsilon(s)|^2 ds \leq C_6(|v_1|^2 + |v_0|^2) + C_7\varepsilon^2 e^{2\beta B(t)} \frac{|v_\varepsilon'(t)|^2}{b(t)} + C_8\varepsilon \int_0^t e^{2\beta B(s)} \frac{|v_\varepsilon'(s)|^2}{b(s)} ds. \tag{4.26}$$

From (4.1) we obtain that

$$\begin{aligned} 2\beta b D_\beta &= 2\beta\varepsilon b e^{2\beta B}\langle v_\varepsilon', v_\varepsilon\rangle + \beta b e^{2\beta B}|v_\varepsilon|^2 \\ &\leq \beta\varepsilon b e^{2\beta B}|v_\varepsilon'|^2 + \beta(1+\varepsilon) b e^{2\beta B}|v_\varepsilon|^2 \\ &\leq \beta\varepsilon b^2 e^{2\beta B}\frac{|v_\varepsilon'|^2}{b} + \frac{\beta}{\sigma_M^2}(1+\varepsilon) b e^{2\beta B}|M^{1/2}v_\varepsilon|^2. \end{aligned} \tag{4.27}$$

From (4.11) and (4.27) we therefore get that

$$D_\beta' + \left(1 - \frac{\beta}{\sigma_M^2}(1+\varepsilon)\right) e^{2\beta B} b|M^{1/2}v_\varepsilon|^2 \leq \varepsilon(b + \beta b^2)e^{2\beta B}\frac{|v_\varepsilon'|^2}{b}. \tag{4.28}$$

Since by (4.2) the function b is bounded then integrating (4.28) we arrive at

$$\left(1 - \frac{\beta}{\sigma_M^2}(1+\varepsilon)\right) \int_0^t e^{2\beta B(s)} b(s)|M^{1/2}v_\varepsilon(s)|^2 ds \leq D_\beta(0) - D_\beta(t) + \varepsilon c_4 \int_0^t e^{2\beta B(s)} \frac{|v_\varepsilon'(s)|^2}{b(s)} ds. \tag{4.29}$$

We can estimate $D_\beta(0)$ and $D_\beta(t)$ as is (4.18) and (4.19); furthermore since $\beta < \sigma_M^2$ we can take ε small enough in such a way that

$$1 - \frac{\beta}{\sigma_M^2}(1+\varepsilon) \geq c_5 > 0. \tag{4.30}$$

Plugging (4.30), (4.18), (4.19) (with β instead of α_0) into (4.29) we achieve (4.26).

Proof of (4.24) *with* $h = 0$. Integrating (4.12) and using (4.26) we get that

$$E_\beta(t) \leq E_\beta(0) - \int_0^t e^{2\beta B(s)} \frac{|v_\varepsilon'(s)|^2}{b(s)} \left(2 + \varepsilon\frac{b'(s)}{b(s)} - 2\beta\varepsilon b(s) - 2\beta\varepsilon C_8\right) ds + 2\beta C_7\varepsilon^2 e^{2\beta B(t)}\frac{|v_\varepsilon'(t)|^2}{b(t)} + 2\beta C_6(|v_1|^2 + |v_0|^2). \tag{4.31}$$

Thanks to (4.2) we can take ε small enough in such a way that

$$2 - 2\varepsilon \sup_{t\geq 0}\frac{|b'(t)|}{b(t)} - 2\beta\varepsilon\sup_{t\geq 0} b(t) - 2\beta\varepsilon C_8 \geq 1, \tag{4.32}$$

$$2\beta C_7\varepsilon \leq \frac{1}{2}. \tag{4.33}$$

Plugging (4.32) and (4.33) into (4.31) we obtain that

$$E_\beta(t) \leq \frac{|v_1|^2}{b(0)} + |M^{1/2}v_0|^2 + 2\beta C_6(|v_1|^2 + |v_0|^2) + \frac{1}{2}E_\beta(t),$$

from which (4.24) immediately follows.

Proof of (4.25). Plugging (4.32) into (4.13) we have that

$$F_\beta' \leq -\frac{1}{\varepsilon} F_\beta + \frac{2}{\varepsilon}\sqrt{F_\beta}|Mv_\varepsilon| e^{\beta B}.$$

Applying (4.24) with $h = 1$ we then obtain that

$$F_\beta' \leq -\frac{1}{\varepsilon}\sqrt{F_\beta}\left(\sqrt{F_\beta} - 2\sqrt{C_4(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)})}\right);$$

hence from Lemma 4.1 we get that

$$F_\beta(t) \leq \max\{F_\beta(0), 4C_4(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)})\} \qquad \forall t \geq 0,$$

that is (4.25).

Proof of (4.10). Since (4.9) holds true, it is enough to prove that (4.8) holds true with $h = 0$ and

$$R_0 = C_9(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)}). \tag{4.34}$$

To this end let us set $\alpha = \sigma_M^2$. By (4.1) we obtain that

$$\begin{aligned} 2\sigma_M^2 b D_{\sigma_M^2} &= 2\sigma_M^2 \varepsilon b e^{2\sigma_M^2 B}\langle v_\varepsilon', v_\varepsilon\rangle + \sigma_M^2 b e^{2\sigma_M^2 B}|v_\varepsilon|^2 \\ &\leq \sigma_M^2 \varepsilon^2 e^{2\sigma_M^2 B}|v_\varepsilon'|^2 + \sigma_M^2 b^2 e^{2\sigma_M^2 B}|v_\varepsilon|^2 + \sigma_M^2 b e^{2\sigma_M^2 B}|v_\varepsilon|^2 \\ &\leq \sigma_M^2 \varepsilon^2 e^{2\sigma_M^2 B}|v_\varepsilon'|^2 + b^2 e^{2\sigma_M^2 B}|M^{1/2}v_\varepsilon|^2 + b e^{2\sigma_M^2 B}|M^{1/2}v_\varepsilon|^2. \end{aligned} \tag{4.35}$$

Moreover from (4.2) and (4.23) we have that

$$e^{2\sigma_M^2 B(t)} = e^{2\beta B(t)} e^{2(\sigma_M^2 - \beta)B(t)} \leq e^{2\beta B(t)} e^{2K_4(\sigma_M^2-\beta)\log(1+t)} = e^{2\beta B(t)}(1+t)^{1/4}; \tag{4.36}$$

hence once again using (4.2) and (4.24) with $h = 0$, inequality (4.35) becomes

$$\begin{aligned} 2\sigma_M^2 b D_{\sigma_M^2} &\leq \sigma_M^2 \varepsilon^2 e^{2\sigma_M^2 B}|v_\varepsilon'|^2 + c_6(1+t)^{-7/4} e^{2\beta B}|M^{1/2}v_\varepsilon|^2 + b e^{2\sigma_M^2 B}|M^{1/2}v_\varepsilon|^2 \\ &\leq \sigma_M^2 \varepsilon^2 e^{2\sigma_M^2 B}|v_\varepsilon'|^2 + c_7(1+t)^{-7/4}(\|v_1\|^2 + |v_0|^2_{D(M^{1/2})}) \\ &\quad + b e^{2\sigma_M^2 B}|M^{1/2}v_\varepsilon|^2. \end{aligned} \tag{4.37}$$

Plugging (4.37) into (4.11) and integrating we obtain that

$$D_{\sigma_M^2}(t) \leq D_{\sigma_M^2}(0) + c_8(\|v_1\|^2 + |v_0|^2_{D(M^{1/2})}) + \varepsilon\int_0^t e^{2\sigma_M^2 B(s)}\frac{|v_\varepsilon'(s)|^2}{b^2(s)} b^2(s)(1+\varepsilon\sigma_M^2)\, ds. \tag{4.38}$$

From (4.36), (4.25), and (4.2) we get that

$$\begin{aligned} e^{2\sigma_M^2 B(t)}\frac{|v_\varepsilon'(t)|^2}{b^2(t)} b^2(t) &\leq e^{2\beta B(t)}\frac{|v_\varepsilon'(t)|^2}{b^2(t)} b^2(t)(1+t)^{1/4} \\ &\leq C_5(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)}) b^2(t)(1+t)^{1/4} \\ &\leq c_9(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)})(1+t)^{-7/4}. \end{aligned} \tag{4.39}$$

Plugging (4.39) into (4.38) we arrive at

$$\begin{aligned}\frac{1}{2}e^{2\sigma_M^2 B(t)}|v_\varepsilon(t)|^2 &\le |D_{\sigma_M^2}(0)| + \varepsilon e^{2\sigma_M^2 B}|\langle v_\varepsilon'(t), v_\varepsilon(t)\rangle| + c_8(\|v_1\|^2 + |v_0|^2_{D(M^{1/2})}) \\ &\quad + \varepsilon c_{10}(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)}) \\ &\le c_{11}(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)}) + \varepsilon^2 e^{2\sigma_M^2 B(t)}\frac{|v_\varepsilon'(t)|^2}{b^2(t)}b^2(t) \\ &\quad + \frac{1}{4}e^{2\sigma_M^2 B(t)}|v_\varepsilon(t)|^2 \\ &\le c_{12}(\|v_1\|^2_{D(M^{1/2})} + |v_0|^2_{D(M)}) + \frac{1}{4}e^{2\sigma_M^2 B(t)}|v_\varepsilon(t)|^2.\end{aligned}$$

By this last inequality (4.34) immediately follows.

Step 3: Proof of (4.6) with $h = 1$. Let $\alpha = \sigma_M^2$. From (4.13), (4.21), and (4.5) used with $h = 1$ we deduce that

$$F_{\sigma_M^2}' \le -\frac{1}{\varepsilon}\sqrt{F_{\sigma_M^2}}\left(\sqrt{F_{\sigma_M^2}} - 2|Mv_\varepsilon|e^{\sigma_M^2 B}\right) \le -\frac{1}{\varepsilon}\sqrt{F_{\sigma_M^2}}\left(\sqrt{F_{\sigma_M^2}} - 2\sqrt{L_{1,M}}\right).$$

We can then apply Lemma 4.1; hence for all $t \ge 0$ we have that

$$F_{\sigma_M^2}(t) \le \max\{F_{\sigma_M^2}(0), 4L_{1,M}\} \le F_{\sigma_M^2}(0) + 4L_{1,M},$$

therefore (4.6) holds true. $\square$

Proof of Proposition 4.3. Let us take ε small enough in such a way that we can apply Proposition 4.2 (with $h = 3$ and $h = 1$).

First let us observe that w_ε satisfies the following problem:

$$\varepsilon w_\varepsilon''(t) + w_\varepsilon'(t) = -b(t)Mv_\varepsilon(t), \quad w_\varepsilon(0) = v_0,\ \ w_\varepsilon'(0) = -b(0)Mv_0,\ \ w_\varepsilon''(0) = 0. \tag{4.40}$$

If we set

$$G(t) := e^{2\sigma_M^2 B(t)}\frac{|w_\varepsilon''(t)|^2}{b^4(t)},$$

therefore from (4.40) we have that

$$G' = G\left(2\sigma_M^2 b - 4\frac{b'}{b}\right) - \frac{2}{\varepsilon}\frac{e^{2\sigma_M^2 B}}{b^4}\langle w_\varepsilon'', -w_\varepsilon'' - bMv_\varepsilon' - b'Mv_\varepsilon\rangle.$$

Hence we immediately get that

$$G' \le -\frac{1}{\varepsilon}G\left(2 - 2\sigma_M^2 b\varepsilon + 4\varepsilon\frac{b'}{b}\right) + \frac{2}{\varepsilon}\sqrt{G}\left(\frac{|Mv_\varepsilon'|}{b} + \frac{|b'|}{b^2}|Mv_\varepsilon|\right)e^{\sigma_M^2 B}. \tag{4.41}$$

Thanks to (4.2) we can take ε small enough so that

$$2 - 2\varepsilon\sigma_M^2 \sup_{t\ge0} b(t) - 4\varepsilon \sup_{t\ge0}\frac{|b'(t)|}{b(t)} \ge 1. \tag{4.42}$$

Using (4.42), (4.2), (4.5) with $h = 1$ and (4.6) with $h = 3$ in (4.41) we obtain that

$$G' \le -\frac{1}{\varepsilon}\sqrt{G}\left(\sqrt{G} - c_1(\|v_1\|^2_{D(M^{3/2})} + |v_0|^2_{D(M^2)})^{1/2}\right),$$

with a constant c_1 that depends only on σ_M^2 and K_3, K_4. Thus we can apply Lemma 4.1, from which we have that

$$G(t) \le \max\{G(0), c_1^2(\|v_1\|^2_{D(M^{3/2})} + |v_0|^2_{D(M^2)})\} \qquad \forall t \ge 0.$$

Since we have that $G(0) = 0$, the thesis is then proved. $\square$

4.2. Proof of Theorem 3.1. We denote by $c_{i,\lambda}$ various constants that depend only on λ and on $|u_0|_{D(A)}$, $|u_1|_{D(A^{1/2})}$.

To begin with let us recall that thanks to (2.5) functions b_ε verify (4.2) independently of ε. Let us also stress that by (2.4) we have $b_\varepsilon(0) = b_0$ independently of ε.

To obtain inequalities (3.1) and (3.2) it is enough to apply Proposition 4.2 with $M = A_\lambda$, $b(t) = b_\varepsilon(t)$, and $\sigma_M^2 = \lambda^2$ (taking of course ε small enough); indeed in such a case $U_{\varepsilon,\lambda}$ solves (4.3).

Now let us prove (3.3).

When the initial data are regular we can directly apply Proposition 4.3 with $M = A_\lambda$ and we obtain (3.3) with a constant that does not depend on ε.

Now let us consider the general case in which $(u_0, u_1) \in D(A) \times D(A^{1/2})$. Let us set

$$\mu^2 := \lambda^2 + \frac{1}{K_3},$$

where K_3 is the constant in (2.5). Then we can write

$$U_{\varepsilon,\lambda} = V_{\varepsilon,\lambda} + U_{\varepsilon,\mu}, \qquad \Theta_{\varepsilon,\lambda} = \theta_{\varepsilon,\lambda} + \Theta_{\varepsilon,\mu}.$$

We estimate separately $V''_{\varepsilon,\lambda} - \theta''_{\varepsilon,\lambda}$ and $U''_{\varepsilon,\mu} - \Theta''_{\varepsilon,\mu}$.

Estimate on $U''_{\varepsilon,\mu} - \Theta''_{\varepsilon,\mu}$. We prove that for every $t \geq 0$ we have that

$$e^{2\lambda^2 B_\varepsilon(t)} \frac{1}{b_\varepsilon^4(t)} |U''_{\varepsilon,\mu}(t) - \Theta''_{\varepsilon,\mu}(t)|^2 \leq \frac{c_{1,\lambda}}{\varepsilon^2}. \tag{4.43}$$

Let us assume that $M = A_\mu$, $b(t) = b_\varepsilon(t)$, and that ε is small enough in such a way that we can apply Proposition 4.2 with these choices. Then from (4.3), (4.5), (4.6) with $h = 1$, and (2.5) we obtain that

$$\begin{aligned} e^{2\lambda^2 B_\varepsilon(t)} |U''_{\varepsilon,\mu}(t)|^2 &\leq \frac{2}{\varepsilon^2} e^{2(\lambda^2-\mu^2)B_\varepsilon(t)} e^{2\mu^2 B_\varepsilon(t)} (b_\varepsilon^2(t)|MU_{\varepsilon,\mu}(t)|^2 + |U'_{\varepsilon,\mu}(t)|^2) \\ &\leq \frac{2b_\varepsilon^2(t)}{\varepsilon^2}(L_0 + L_1)(\|u_1\|^2_{D(A^{1/2})} + |u_0|^2_{D(A)}) e^{2(\lambda^2-\mu^2)K_3 \log(1+t)} \\ &= \frac{1}{\varepsilon^2} c_{2,\lambda} b_\varepsilon^2(t)(1+t)^{-2}. \end{aligned} \tag{4.44}$$

Moreover $\Theta_{\varepsilon,\mu}$ verifies (2.11), thence from (2.5) it follows that

$$\begin{aligned} e^{2\lambda^2 B_\varepsilon(t)} |\Theta''_{\varepsilon,\mu}(t)|^2 &\leq \frac{1}{\varepsilon^2} |\Theta'_{\varepsilon,\mu}(0)|^2 e^{-2t/\varepsilon} e^{2\lambda^2 B_\varepsilon(t)} \\ &\leq \frac{c_{3,\lambda}}{\varepsilon^2} e^{-2t/\varepsilon} e^{2\lambda^2 K_4 \log(1+t)}. \end{aligned} \tag{4.45}$$

Using (4.44), (4.45), and (2.5) we get that

$$\begin{aligned} \frac{e^{2\lambda^2 B_\varepsilon(t)}}{b_\varepsilon^4(t)} |U''_{\varepsilon,\mu}(t) - \Theta''_{\varepsilon,\mu}(t)|^2 &\leq \frac{2}{b_\varepsilon^4(t)} e^{2\lambda^2 B_\varepsilon(t)} (|U''_{\varepsilon,\mu}(t)|^2 + |\Theta''_{\varepsilon,\mu}(t)|^2) \\ &\leq \frac{2c_{2,\lambda}}{\varepsilon^2 b_\varepsilon^2(t)}(1+t)^{-2} + \frac{2c_{3,\lambda}}{\varepsilon^2 b_\varepsilon^4(t)} e^{-2t/\varepsilon} e^{2\lambda^2 K_4 \log(1+t)} \\ &\leq \frac{1}{\varepsilon^2}\left[c_{4,\lambda} + c_{5,\lambda}(1+t)^4 e^{-t} e^{2\lambda^2 K_4 \log(1+t)}\right] \leq \frac{c_{6,\lambda}}{\varepsilon^2}, \end{aligned}$$

that is (4.43).

Estimate on $V''_{\varepsilon,\lambda}-\theta''_{\varepsilon,\lambda}$. Let $M=A_{[\lambda,\mu)}$ and $b(t)=b_\varepsilon(t)$. Then $V_{\varepsilon,\lambda}$ and $\theta_{\varepsilon,\lambda}$ are the solutions of the corresponding problems (4.3) and (4.4). Moreover since $A_{[\lambda,\mu)}$ is a bounded operator we have that the relative initial data $(v_0,v_1)\in D(M^2)\times D(M^{3/2})$ and

$$|v_0|^2_{D(M^2)}+|v_1|^2_{D(M^{3/2})}\leq c_{7,\lambda}.$$

Let ε small in such a way that we can apply Proposition 4.3 with these choices. Then since $w''_\varepsilon=V''_{\varepsilon,\lambda}-\theta''_{\varepsilon,\lambda}$, and $\sigma_M^2=\lambda^2$, we have that for every $t\geq 0$

$$e^{2\lambda^2 B_\varepsilon(t)}\frac{1}{b_\varepsilon^4(t)}|V''_{\varepsilon,\lambda}(t)-\theta''_{\varepsilon,\lambda}(t)|^2\leq c_{8,\lambda}. \tag{4.46}$$

Conclusion. The inequality (3.3) in the general case is a straightforward consequence of (4.43) and (4.46). □

4.3. A decomposition of u_ε. Let u_ε be the solution of (1.1)–(1.2) as in Theorem 2.1 and let $u_{\varepsilon,\nu}$ be defined as in (2.10). Moreover let us set

$$|A^{1/2}u_\varepsilon(t)|^2=\nu^2|u_{\varepsilon,\nu}(t)|^2+\alpha_{\varepsilon,1}(t),\qquad |Au_\varepsilon(t)|^2=\nu^4|u_{\varepsilon,\nu}(t)|^2+\alpha_{\varepsilon,2}(t), \tag{4.47}$$

$$\frac{|u'_\varepsilon(t)|^2}{b_\varepsilon^2(t)}=\frac{|u'_{\varepsilon,\nu}(t)|^2}{b_\varepsilon^2(t)}+\alpha_{\varepsilon,3}(t),\quad \frac{|A^{1/2}u'_\varepsilon(t)|^2}{b_\varepsilon^2(t)}=\frac{\nu^2|u'_{\varepsilon,\nu}(t)|^2}{b_\varepsilon^2(t)}+\alpha_{\varepsilon,4}(t), \tag{4.48}$$

and

$$e^{2\nu^2 B_\varepsilon(t)}|u_{\varepsilon,\nu}|^2=\beta_{\varepsilon,0}(t),\quad e^{2\nu^2 B_\varepsilon(t)}\alpha_{\varepsilon,1}(t)=\beta_{\varepsilon,1}(t),\quad e^{2\nu^2 B_\varepsilon(t)}\alpha_{\varepsilon,2}(t)=\beta_{\varepsilon,2}(t), \tag{4.49}$$

$$e^{2\nu^2 B_\varepsilon(t)}\alpha_{\varepsilon,3}(t)=\beta_{\varepsilon,3}(t),\qquad e^{2\nu^2 B_\varepsilon(t)}\alpha_{\varepsilon,4}(t)=\beta_{\varepsilon,4}(t). \tag{4.50}$$

In the proposition below we study the behavior of the quantities defined in (4.49)–(4.50).

PROPOSITION 4.4. *For ε small enough the following properties hold true.*

1. *For $t\to+\infty$ we have that*

$$\beta_{\varepsilon,1}(t)\to 0,\qquad \beta_{\varepsilon,2}(t)\to 0,\qquad \beta_{\varepsilon,3}(t)\to 0,\qquad \beta_{\varepsilon,4}(t)\to 0, \tag{4.51}$$

$$\beta_{\varepsilon,0}(t)\to L_\varepsilon\in\mathbb{R}\setminus\{0\}. \tag{4.52}$$

2. *If $u_{0,\nu}\neq 0$ then there exists a constant $K_7>0$ independent of ε and t such that*

$$\beta_{\varepsilon,0}(t)\geq K_7\qquad \forall t\geq 0. \tag{4.53}$$

Proof of Proposition 4.4. Let us denote by c_i various constants that depend only on ν, $|u_0|_{D(A)}$ and $|u_1|_{D(A^{1/2})}$.

Proof of (4.51). Let us choose

$$\delta^2:=\nu^2+\frac{1}{K_3}.$$

Let us assume that ε is small enough so that we can use Theorem 3.1 with $\lambda=\delta$.

We can rewrite the quantities in (4.47) and (4.48) as

$$\alpha_{\varepsilon,h}(t)=\sum_{k:\nu<\lambda_k<\delta}\lambda_k^{2h}|u_{\varepsilon,k}(t)|^2+|A^{h/2}U_{\varepsilon,\delta}(t)|^2=\alpha_{\varepsilon,h,1}(t)+\alpha_{\varepsilon,h,2}(t)\quad\text{for }\ h=1,2,$$

and for $h = 3, 4$,

$$\alpha_{\varepsilon,h}(t) = \frac{1}{b_\varepsilon^2(t)} \left(\sum_{k:\nu<\lambda_k<\delta} \lambda_k^{2(h-3)} |u'_{\varepsilon,k}(t)|^2 + |A^{(h-3)/2} U'_{\varepsilon,\delta}(t)|^2 \right) = \alpha_{\varepsilon,h,1}(t) + \alpha_{\varepsilon,h,2}(t).$$

Since it holds true that

$$\frac{|A^{h/2} U'_{\varepsilon,\delta}(t)|^2}{b_\varepsilon^2(t)} = \frac{\varepsilon |A^{h/2} U'_{\varepsilon,\delta}(t)|^2}{b_\varepsilon(t)} \frac{1}{\varepsilon} \frac{1}{b_\varepsilon(t)},$$

thence, thanks to (3.1) with $h = 0$ and $h = 1$ and (2.5) we have that

$$\begin{aligned} &e^{2\nu^2 B_\varepsilon(t)} (\alpha_{\varepsilon,1,2}(t) + \alpha_{\varepsilon,2,2}(t) + \alpha_{\varepsilon,3,2}(t) + \alpha_{\varepsilon,4,2}(t)) \\ &\quad \leq \frac{c_1}{\varepsilon} \frac{1}{b_\varepsilon(t)} e^{2(\nu^2-\delta^2) B_\varepsilon(t)} \leq \frac{c_2}{\varepsilon} (1+t) e^{-2\log(1+t)}. \end{aligned}$$

Hence we get that

$$\lim_{t\to+\infty} e^{2\nu^2 B_\varepsilon(t)} (\alpha_{\varepsilon,1,2}(t) + \alpha_{\varepsilon,2,2}(t) + \alpha_{\varepsilon,3,2}(t) + \alpha_{\varepsilon,4,2}(t)) = 0. \tag{4.54}$$

For $\nu < \lambda_k < \delta$ let us now consider $b(t) = b_\varepsilon(t)$ and $M = A_{\{\lambda_k\}}$. Then from (2.5) the function b verifies (4.2) and M verifies (4.1) with $\sigma_M^2 = \lambda_k^2$. Let ε small enough so that we can apply Proposition 4.2 with such choices. We stress that since $\nu \leq \lambda_k \leq \delta$ we can take the smallness of ε independently of λ_k. Since $\lambda_k > \nu$ and $\lambda_k < \delta$, moreover, from (4.5) and (4.6) (with $h = 1$) we have that

$$\begin{aligned} &e^{2\nu^2 B_\varepsilon(t)} (\alpha_{\varepsilon,1,1}(t) + \alpha_{\varepsilon,2,1}(t) + \alpha_{\varepsilon,3,1}(t) + \alpha_{\varepsilon,4,1}(t)) \\ &\quad \leq c_3 \sum_{k:\nu<\lambda_k<\delta} \left(2\lambda_k^4 |u_{\varepsilon,k}(t)|^2 + 2\frac{|u'_{\varepsilon,k}(t)|^2}{b_\varepsilon^2(t)} \right) e^{2\lambda_k^2 B_\varepsilon(t)} e^{2(\nu^2-\lambda_k^2) B_\varepsilon(t)} \\ &\quad \leq c_4 \sum_{k:\nu<\lambda_k<\delta} (\lambda_k^2 + \lambda_k^4 + 1)(|u_{0,k}|^2 + |u_{1,k}|^2) e^{2(\nu^2-\lambda_k^2) B_\varepsilon(t)} \\ &\quad \leq c_5 \sum_{k:\nu<\lambda_k<\delta} (|u_{0,k}|^2 + |u_{1,k}|^2) e^{2(\nu^2-\lambda_k^2) B_\varepsilon(t)}, \end{aligned} \tag{4.55}$$

we can therefore pass to the limit in (4.55) so that

$$\begin{aligned} 0 &\leq \lim_{t\to+\infty} e^{2\nu^2 B_\varepsilon(t)} (\alpha_{\varepsilon,1,1}(t) + \alpha_{\varepsilon,2,1}(t) + \alpha_{\varepsilon,3,1}(t) + \alpha_{\varepsilon,4,1}(t)) \\ &\leq c_5 \sum_{k:\nu<\lambda_k<\delta} \lim_{t\to+\infty} (|u_{0,k}|^2 + |u_{1,k}|^2) e^{2(\nu^2-\lambda_k^2) B_\varepsilon(t)} = 0. \end{aligned} \tag{4.56}$$

From (4.54) and (4.56) we immediately get (4.51).

Proof of (4.52)–(4.53). Let us set $y_\varepsilon(t) := |u_{\varepsilon,\nu}(t)|^2$. Then y_ε solves

$$y'_\varepsilon = -2\nu^2 b_\varepsilon y_\varepsilon - 2\varepsilon \langle u_{\varepsilon,\nu}, u''_{\varepsilon,\nu} \rangle; \tag{4.57}$$

thence for all $t \geq 0$ we get that

$$e^{2\nu^2 B_\varepsilon(t)} y_\varepsilon(t) = |u_{0,\nu}|^2 - 2\varepsilon \int_0^t \langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s) \rangle e^{2\nu^2 B_\varepsilon(s)} ds. \tag{4.58}$$

Let us now estimate $\langle u_{\varepsilon,\nu}, u''_{\varepsilon,\nu}\rangle$. Let us choose $M = A_{\{\nu\}}$, $b(t) = b_\varepsilon(t)$, and let ε small so that we can apply Propositions 4.2 and 4.3 (with $v_\varepsilon = u_{\varepsilon,\nu}$). This is possible since in such a case

$$|u_{1,\nu}|^2_{D(M^{3/2})} + |u_{0,\nu}|^2_{D(M^2)} \le c_6(|u_{1,\nu}|^2 + |u_{0,\nu}|^2).$$

Moreover clearly we have that

$$|u_{\varepsilon,\nu}(t)|^2 = \nu^{-4}|Mu_{\varepsilon,\nu}(t)|^2.$$

Then from (4.5) with $h = 1$ and (4.7), using (2.5) (or equivalently (4.2)), we obtain that

$$\begin{aligned}
|\langle u_{\varepsilon,\nu}(t), u''_{\varepsilon,\nu}(t)\rangle| e^{2\nu^2 B_\varepsilon(t)} &\le |u_{\varepsilon,\nu}(t)|(|w''_\varepsilon(t)| + |\theta''_\varepsilon(t)|)e^{2\nu^2 B_\varepsilon(t)} \\
&\le c_7(\|u_{1,\nu}\| + |u_{0,\nu}|)\left(\frac{|w''_\varepsilon(t)|}{b^2_\varepsilon(t)} b^2_\varepsilon(t) + \frac{1}{\varepsilon}|\theta'_\varepsilon(0)|e^{-t/\varepsilon}\right) e^{\nu^2 B_\varepsilon(t)} \\
&\le c_8(\|u_{1,\nu}\|^2 + |u_{0,\nu}|^2)\left(b^2_\varepsilon(t) + \frac{1}{\varepsilon}e^{-t/\varepsilon}e^{\nu^2 B_\varepsilon(t)}\right) \\
&\le c_9(\|u_{1,\nu}\|^2 + |u_{0,\nu}|^2)\left(\frac{1}{(1+t)^2} + \frac{1}{\varepsilon}e^{-t/\varepsilon}e^{\nu^2 K_4 \log(1+t)}\right).
\end{aligned}$$

Using that

$$\sup_{t\ge 0} e^{-t/2} e^{\nu^2 K_4 \log(1+t)} < +\infty,$$

we arrive at

$$|\langle u_{\varepsilon,\nu}(t), u''_{\varepsilon,\nu}(t)\rangle| e^{2\nu^2 B_\varepsilon(t)} \le c_{10}(\|u_{1,\nu}\|^2 + |u_{0,\nu}|^2)\left(\frac{1}{(1+t)^2} + \frac{1}{\varepsilon}e^{-t/2\varepsilon}\right). \tag{4.59}$$

From (4.59) we thus get for all $t \ge 0$ that

$$\begin{aligned}
\left|\int_0^t \langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s)\rangle e^{2\nu^2 B_\varepsilon(s)} ds\right| &\le \int_0^t |\langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s)\rangle| e^{2\nu^2 B_\varepsilon(s)} ds \\
&\le c_{11}(\|u_{1,\nu}\|^2 + |u_{0,\nu}|^2),
\end{aligned} \tag{4.60}$$

and also

$$\lim_{t\to+\infty} \int_0^t \langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s)\rangle e^{2\nu^2 B_\varepsilon(s)} ds = S_\varepsilon \in \mathbb{R}. \tag{4.61}$$

Therefore from (4.58) and (4.61) we have that there exists

$$\lim_{t\to+\infty} e^{2\nu^2 B_\varepsilon(t)} y_\varepsilon(t) = |u_{0,\nu}|^2 - 2\varepsilon S_\varepsilon.$$

We have to prove that this limit is not zero.

Case $u_{0,\nu} \ne 0$. By (4.60), for ε small we have that

$$2\varepsilon \left|\int_0^t \langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s)\rangle e^{2\nu^2 B_\varepsilon(s)} ds\right| \le \frac{1}{2}|u_{0,\nu}|^2 \qquad \forall t \ge 0,$$

and hence (4.53) follows from (4.58) and, as a consequence, the limit in (4.52) is different from zero.

Case $u_{0,\nu} = 0$. Since $u_{1,\nu} \neq 0$, then there exists a single *real* component of $u_{1,\nu}$ different from zero, that we indicate by $u_{1,\nu,r}$. Let $u_{\varepsilon,\nu,r}$ be the relative component of $u_{\varepsilon,\nu}$. We prove that

$$\lim_{t\to+\infty} e^{2\nu^2 B_\varepsilon(t)}|u_{\varepsilon,\nu,r}(t)|^2 \neq 0. \tag{4.62}$$

This will be enough to prove that the limit in (4.52) is not zero.

To begin with, let us remark that there exists $T_\varepsilon > 0$ such that

$$u'_{\varepsilon,\nu,r}(T_\varepsilon) = 0.$$

Indeed if it is not the case, then $u_{\varepsilon,\nu,r}$ is a strictly increasing or decreasing function and since $u_{\varepsilon,\nu,r}(0) = 0$, we therefore get that

$$\lim_{t\to+\infty} u_{\varepsilon,\nu,r}(t) \neq 0,$$

but this is in contrast with (2.1), since $|A^{1/2}u_\varepsilon(t)|^2 \geq \nu^2|u_{\varepsilon,\nu,r}(t)|^2$ for all $t \geq 0$.

Let now us set

$$T_{\varepsilon,0} := \sup\{\tau \geq 0 :\, u'_{\varepsilon,\nu,r}(t) \neq 0,\ \forall t \in [0,\tau]\}.$$

As seen before, $T_{\varepsilon,0}$ is a real positive number, and moreover

$$u'_{\varepsilon,\nu,r}(T_{\varepsilon,0}) = 0,$$

and in $[0, T_{\varepsilon,0}[$ the function $u_{\varepsilon,\nu,r}$ is strictly increasing or decreasing, so that

$$u_{\varepsilon,\nu,r}(T_{\varepsilon,0}) = P_\varepsilon \neq 0.$$

Therefore, as in (4.57)–(4.58) for $t \geq T_{\varepsilon,0}$ we have that

$$e^{2\nu^2(B_\varepsilon(t)-B_\varepsilon(T_{\varepsilon,0}))}|u_{\varepsilon,\nu,r}(t)|^2 = P_\varepsilon^2 - 2\varepsilon\int_{T_{\varepsilon,0}}^t u_{\varepsilon,\nu,r}(s)u''_{\varepsilon,\nu,r}(s)e^{2\nu^2(B_\varepsilon(s)-B_\varepsilon(T_{\varepsilon,0}))}ds. \tag{4.63}$$

Now for $t \geq 0$, let us set $v_\varepsilon(t) = u_{\varepsilon,\nu,r}(t+T_{\varepsilon,0})$. Then v_ε verifies (4.3) with $M = A_{\{\nu\}}$ restricted to the single component $u_{\varepsilon,\nu,r}$, $b(t) = b_\varepsilon(t+T_{\varepsilon,0})$ and initial data $v_\varepsilon(0) = P_\varepsilon$, $v'_\varepsilon(0) = 0$. Thanks to (2.5) it is clear that the function b verifies (4.2). Therefore we can obtain, as in (4.59) and (4.60),

$$\left|\int_{T_{\varepsilon,0}}^t u_{\varepsilon,\nu,r}(s)u''_{\varepsilon,\nu,r}(s)e^{2\nu^2(B_\varepsilon(s)-B_\varepsilon(T_{\varepsilon,0}))}ds\right| \leq c_{11}P_\varepsilon^2 \qquad \forall t \geq T_{\varepsilon,0}. \tag{4.64}$$

Only we have to specify that

$$\sup_{t\geq T_{\varepsilon,0}} e^{-(t-T_{\varepsilon,0})/2}e^{\nu^2K_4(\log(1+t)-\log(1+T_{\varepsilon,0}))} \leq \sup_{t\geq 0} e^{-t/2}e^{\nu^2K_4\log(1+t)} < +\infty.$$

Let now ε be small enough so that $\varepsilon c_{11} \leq 1/2$, then from (4.63) and (4.64) we get that

$$e^{2\nu^2(B_\varepsilon(t)-B_\varepsilon(T_{\varepsilon,0}))}|u_{\varepsilon,\nu,r}(t)|^2 \geq \frac{1}{2}P_\varepsilon^2 \qquad \forall t \geq T_{\varepsilon,0},$$

and thus the limit in (4.62) is different from zero. $\square$

4.4. Proof of Theorem 3.3. Let us assume that ε is small enough so that Theorem 3.1 with $\lambda = \nu$ and Proposition 4.4 hold true. Let us moreover denote by c_i various constants that depend only on ν, $|u_0|_{D(A)}$, and $|u_1|_{D(A^{1/2})}$ and by $c_{i,\varepsilon}$ constants that depend also on ε.

Proof of (3.5). Since the limit in (4.52) is different from zero, there exists $T_{\varepsilon,1} \geq 0$ such that

$$\beta_{\varepsilon,0}(t) \geq c_{1,\varepsilon} > 0 \qquad \forall t \geq T_{\varepsilon,1}$$

and in particular

$$|u_{\varepsilon,\nu}(t)| > 0 \qquad \forall t \geq T_{\varepsilon,1}.$$

Let us remark that if $u_{0,\nu} \neq 0$, then thanks to (4.53) we can take $T_{\varepsilon,1} = 0$.

Thanks to (4.47) and (2.4) for $t \geq T_{\varepsilon,1}$ we have that

$$\begin{aligned} b_\varepsilon(t)e^{2\nu^2\gamma B_\varepsilon(t)} &= \left(\nu^2|u_{\varepsilon,\nu}(t)|^2 + \alpha_{\varepsilon,1}(t)\right)^\gamma e^{2\nu^2\gamma B_\varepsilon(t)} \\ &= \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t)\left(1 + \frac{\alpha_{\varepsilon,1}(t)}{\nu^2|u_{\varepsilon,\nu}(t)|^2}\right)^\gamma \\ &= \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t)\left(1 + \frac{\beta_{\varepsilon,1}(t)}{\nu^2\beta_{\varepsilon,0}(t)}\right)^\gamma. \end{aligned} \tag{4.65}$$

Since for all $x \geq 0$ there exists $0 \leq \xi \leq x$ such that

$$(1+x)^\gamma = 1 + \gamma(1+\xi)^{\gamma-1}x,$$

then for $t \geq T_{\varepsilon,1}$ we can rewrite (4.65) as

$$b_\varepsilon(t)e^{2\nu^2\gamma B_\varepsilon(t)} = \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t)\left(1 + \gamma(1+\xi)^{\gamma-1}\frac{\beta_{\varepsilon,1}(t)}{\nu^2\beta_{\varepsilon,0}(t)}\right) =: \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t) + \phi_\varepsilon(t), \tag{4.66}$$

where if $\gamma < 1$ then

$$0 \leq \phi_\varepsilon(t) \leq c_1\beta_{\varepsilon,0}^{\gamma-1}(t)\beta_{\varepsilon,1}(t), \tag{4.67}$$

while if $\gamma \geq 1$ then

$$0 \leq \phi_\varepsilon(t) \leq c_2\beta_{\varepsilon,0}^\gamma(t)\left(1 + \frac{\beta_{\varepsilon,1}(t)}{\beta_{\varepsilon,0}(t)}\right)^{\gamma-1}\frac{\beta_{\varepsilon,1}(t)}{\beta_{\varepsilon,0}(t)} \leq c_2\left(\beta_{\varepsilon,0}(t) + \beta_{\varepsilon,1}(t)\right)^{\gamma-1}\beta_{\varepsilon,1}(t). \tag{4.68}$$

Integrating (4.66) we get that

$$e^{2\nu^2\gamma B_\varepsilon(t)} - e^{2\nu^2\gamma B_\varepsilon(T_{\varepsilon,1})} = 2\nu^2\gamma\left[\int_{T_{\varepsilon,1}}^t \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(s) + \phi_\varepsilon(s)\,ds\right] \qquad \forall t \geq T_{\varepsilon,1}. \tag{4.69}$$

From (4.52), (4.51), and (4.67) or (4.68) we immediately obtain that

$$\lim_{t\to+\infty}\frac{1}{1+t}\int_{T_1}^t \beta_{\varepsilon,0}^\gamma(s)\,ds = L_\varepsilon^\gamma, \tag{4.70}$$

$$\lim_{t\to+\infty}\phi_\varepsilon(t) = 0, \tag{4.71}$$

$$\lim_{t\to+\infty}\frac{1}{1+t}\int_{T_1}^t \phi_\varepsilon(s)\,ds = 0. \tag{4.72}$$

From (4.69), (4.70), and (4.72) we then deduce that

$$\lim_{t\to+\infty} \frac{1}{1+t} e^{2\nu^2\gamma B_\varepsilon(t)} = 2\nu^{2(\gamma+1)}\gamma L_\varepsilon^\gamma. \tag{4.73}$$

This nonzero limit proves (3.5) with constants depending on ε.

Let us now assume that $u_{0,\nu} \neq 0$ so that $T_{\varepsilon,1} = 0$ and (4.53) holds true. Then from (4.69) we obtain that

$$e^{2\nu^2\gamma B_\varepsilon(t)} \geq 1 + 2\nu^{2(\gamma+1)}\gamma K_7^\gamma t \qquad \forall t \geq 0. \tag{4.74}$$

Moreover since $u_\varepsilon = U_{\varepsilon,\nu}$, from (3.1) of Theorem 3.1 (with $h = 0$ and $\lambda = \nu$) we get that

$$b_\varepsilon(t) e^{2\nu^2\gamma B_\varepsilon(t)} = (e^{2\nu^2 B_\varepsilon(t)} |A^{1/2}u_\varepsilon(t)|^2)^\gamma \leq \gamma_{0,\nu}^\gamma$$

and hence for all $t \geq 0$ we have that

$$e^{2\nu^2\gamma B_\varepsilon(t)} = 1 + 2\nu^2\gamma \int_0^t (e^{2\nu^2 B_\varepsilon(s)}|A^{1/2}u_\varepsilon(s)|^2)^\gamma ds \leq 1 + 2\nu^2\gamma\, \gamma_{0,\nu}^\gamma t. \tag{4.75}$$

Thus from (4.74) and (4.75) we obtain (3.5) with constants independent of ε.

Proof of (3.6). From (4.69), for $t \geq T_{\varepsilon,1}$ we have that

$$B_\varepsilon(t) = \frac{1}{2\nu^2\gamma} \log\left(e^{2\nu^2\gamma B_\varepsilon(T_{\varepsilon,1})} + 2\nu^2\gamma \int_{T_{\varepsilon,1}}^t \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(s) + \phi_\varepsilon(s)\, ds \right). \tag{4.76}$$

Taking the derivative of (4.76) we obtain that

$$b_\varepsilon(t) = \frac{\nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t) + \phi_\varepsilon(t)}{e^{2\nu^2\gamma B_\varepsilon(T_{\varepsilon,1})} + 2\nu^2\gamma \displaystyle\int_{T_{\varepsilon,1}}^t \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(s) + \phi_\varepsilon(s)\, ds}.$$

Using (4.70), (4.71), (4.72), and (4.52) we get that

$$\begin{aligned} &\lim_{t\to+\infty} (1+t) b_\varepsilon(t) \\ &\quad = \lim_{t\to+\infty} \frac{\nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t) + \phi_\varepsilon(t)}{\dfrac{1}{1+t}\left[e^{2\nu^2\gamma B_\varepsilon(T_{\varepsilon,1})} + 2\nu^2\gamma \displaystyle\int_{T_{\varepsilon,1}}^t \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(s) + \phi_\varepsilon(s)\, ds \right]} = \frac{1}{2\nu^2\gamma}, \end{aligned}$$

that is (3.6).

Limit of $|A^{1/2}u_\varepsilon|$. We prove that

$$\lim_{t\to+\infty} (1+t)^{1/\gamma}|A^{1/2}u_\varepsilon(t)|^2 = \frac{1}{(2\nu^2\gamma)^{1/\gamma}}. \tag{4.77}$$

To this end it is enough to remark that

$$(1+t)^{1/\gamma}|A^{1/2}u_\varepsilon(t)|^2 = ((1+t)b_\varepsilon(t))^{1/\gamma}$$

and use (3.6).

Proof of (3.7). From (4.47) we have that

$$\begin{aligned} (1+t)^{1/\gamma}\nu^2|u_{\varepsilon,\nu}(t)|^2 &= (1+t)^{1/\gamma}|A^{1/2}u_\varepsilon(t)|^2 - (1+t)^{1/\gamma}\alpha_{\varepsilon,1}(t) \\ &= (1+t)^{1/\gamma}|A^{1/2}u_\varepsilon(t)|^2 - (1+t)^{1/\gamma}e^{-2\nu^2 B_\varepsilon(t)}\beta_{\varepsilon,1}(t). \end{aligned} \tag{4.78}$$

From (3.5) we know that

$$(1+t)^{1/\gamma}e^{-2\nu^2 B_\varepsilon(t)} \leq c_{2,\varepsilon} \qquad \forall t \geq 0, \tag{4.79}$$

and hence from (4.77), (4.51), and (4.78) we deduce that

$$\lim_{t\to+\infty}(1+t)^{1/\gamma}\nu^2|u_{\varepsilon,\nu}(t)|^2 = \frac{1}{(2\nu^2\gamma)^{1/\gamma}}, \tag{4.80}$$

whence (3.7) immediately follows.

Proof of (3.8). Thanks to (3.3) in Theorem 3.1 we have for all $t \geq 0$ that

$$|U''_{\varepsilon,\nu}(t)| \leq |U''_{\varepsilon,\nu}(t) - \Theta''_{\varepsilon,\nu}(t)| + |\Theta''_{\varepsilon,\nu}(t)| \leq \sqrt{\gamma_{\varepsilon,\nu}}\, b_\varepsilon^2(t)e^{-\nu^2 B_\varepsilon(t)} + \frac{1}{\varepsilon}|\Theta'_{\varepsilon,\nu}(0)|e^{-t/\varepsilon}. \tag{4.81}$$

Using (4.79) and (2.5) in (4.81) we get that

$$|U''_{\varepsilon,\nu}(t)|^2 \leq c_{3,\varepsilon}\left(\frac{1}{(1+t)^{2+1/(2\gamma)}} + e^{-t}\right)^2 \leq c_{4,\varepsilon}\frac{1}{(1+t)^{4+1/\gamma}}, \tag{4.82}$$

that is (3.8), since $u_\varepsilon = U_{\varepsilon,\nu}$.

Existence of $u_{\varepsilon,\infty}$. Thanks to (2.9), (4.79), (4.47), (4.49), and (2.5) we have that for all $t \geq 0$

$$(1+t)^{1/\gamma}|u_\varepsilon(t) - u_{\varepsilon,\nu}(t)|^2_{D(A)} \leq c_3(1+t)^{1/\gamma}|Au_\varepsilon(t) - Au_{\varepsilon,\nu}(t)|^2 \leq c_{4,\varepsilon}\beta_{\varepsilon,2}(t),$$

$$(1+t)^{2+1/\gamma}|u'_\varepsilon(t) - u'_{\varepsilon,\nu}(t)|^2_{D(A^{1/2})} \leq c_{5,\varepsilon}(1+t)^2 b_\varepsilon^2(t)\beta_{\varepsilon,4}(t) \leq c_{6,\varepsilon}\beta_{\varepsilon,4}(t),$$

and hence from (4.51) we obtain that

$$\lim_{t\to+\infty}(1+t)^{1/\gamma}|u_\varepsilon(t) - u_{\varepsilon,\nu}(t)|^2_{D(A)} + (1+t)^{2+1/\gamma}|u'_\varepsilon(t) - u'_{\varepsilon,\nu}(t)|^2_{D(A^{1/2})} = 0.$$

Therefore for proving (3.4) we have only to show that the functions $(1+t)^{1/(2\gamma)}u_{\varepsilon,\nu}(t)$ and $(1+t)^{1+1/(2\gamma)}u'_{\varepsilon,\nu}(t)$ have the required limits. Since

$$u'_{\varepsilon,\nu}(t) = -\nu^2 b_\varepsilon(t)u_{\varepsilon,\nu}(t) - \varepsilon u''_{\varepsilon,\nu}(t),$$

then we have that

$$e^{\nu^2 B_\varepsilon(t)}u_{\varepsilon,\nu}(t) = u_{0,\nu} - \varepsilon\int_0^t e^{\nu^2 B_\varepsilon(s)}u''_{\varepsilon,\nu}(s)\,ds.$$

Thanks to (3.8) it is clear that for all $t \geq 0$

$$|u''_{\varepsilon,\nu}(t)| \leq |u''_\varepsilon(t)| \leq \sqrt{K_\varepsilon}\frac{1}{(1+t)^{2+1/(2\gamma)}}, \tag{4.83}$$

thus using once again (3.5) we obtain that there exists

$$\lim_{t\to+\infty}\int_0^t e^{\nu^2 B_\varepsilon(s)}u''_{\varepsilon,\nu}(s)\,ds = \alpha_{\varepsilon,\nu} \in H_{\{\nu\}}.$$

Applying (4.73) we finally arrive at

$$\begin{aligned}\lim_{t\to+\infty}(1+t)^{1/(2\gamma)}u_{\varepsilon,\nu}(t) &= \lim_{t\to+\infty}(1+t)^{1/(2\gamma)}e^{-\nu^2 B_\varepsilon(t)}e^{\nu^2 B_\varepsilon(t)}u_{\varepsilon,\nu}(t)\\ &= \frac{1}{\left(2\nu^{2(\gamma+1)}\gamma L_\varepsilon^\gamma\right)^{1/(2\gamma)}}(u_{0,\nu} - \varepsilon\alpha_{\varepsilon,\nu}) = u_{\varepsilon,\infty} \in H_{\{\nu\}}.\end{aligned} \tag{4.84}$$

Furthermore we have also that

$$(1+t)^{1+1/(2\gamma)}u'_{\varepsilon,\nu}(t) = -\nu^2(1+t)b_\varepsilon(t)(1+t)^{1/(2\gamma)}u_{\varepsilon,\nu}(t) - \varepsilon(1+t)^{1+1/(2\gamma)}u''_{\varepsilon,\nu}(t),$$

therefore from (3.6), (4.84), and (4.83) we get that

$$\lim_{t\to+\infty}(1+t)^{1+1/(2\gamma)}u'_{\varepsilon,\nu}(t) = -\frac{1}{2\gamma}u_{\varepsilon,\infty} \in H_{\{\nu\}}. \tag{4.85}$$

From (4.84), (4.85), and (3.7) the existence of the required nonzero limits follows. □

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